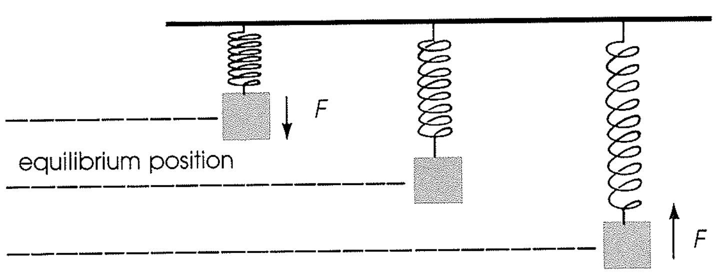
Practice

1. Determine the frequency for each of the following:
   (a) A bouncing spring completes 10 vibrations in 7.6 s.
   (b) An atom vibrates 2.5×10^{10} times in 5.0 s.
   (c) A sound wave from a guitar string has a period of 3.3×10^{-3} s. (1.3 Hz, 5.0×10^{9} Hz, 3.0×10^{2} Hz)
2. Find the period for each of the following:
   (a) A pendulum swings back and forth 20 times in 15 s.
   (b) A light wave has a frequency of 5.0×10^{14} Hz.
   (c) The moon travels around the Earth six times in 163.8 d.
   (0.75 s, 2.0×10^{-15} s, 27.3 d)

12.2 Simple Harmonic Motion

As a mass hanging from an ordinary spring bounces up and down, we see that its speed is greatest when it is at the midpoint, whether it is going up or down. We see also that it slows down at both ends of its path, momentarily stopping before it reverses direction. This is because, except at the equilibrium position where the net force is zero, there is always an unbalanced force that acts to restore the mass to the equilibrium position.



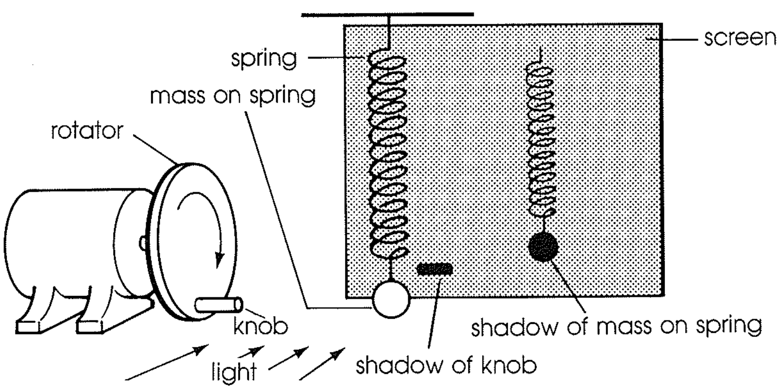
As the mass vibrates up and down, $F \propto x$ and $a \propto x$. These are the conditions that define simple harmonic motion.

Assuming the ideal condition of a massless spring, the mass moves the same distance above the equilibrium position as it moves below, completing each half-cycle in the same amount of time. At every point, including the equilibrium position, the net force exerted on the mass, and therefore the resulting acceleration, is proportional to the displacement of the mass from the equilibrium position. Both the force and the acceleration act towards the equilibrium position. This type of oscillation, where the restoring force is proportional to the displacement, is known as a **simple harmonic motion (S.H.M.)**. In other words, for simple harmonic motion, $F \propto x$.

You will note that this takes the form of Hooke's Law, that is, $F = kx$ (Section 6.7). Thus a loaded spring or flexible beam, a stretched wire, a twisted steel rod, a vibrating guitar string — any elastic system obeying Hooke's Law — may oscillate with simple harmonic motion. There are many other phenomena with characteristics resembling simple harmonic motion. This is why the analysis of simple harmonic motion is widely used in physics.

Period of Simple Harmonic Motion — The Reference Circle

As illustrated, alongside a ball bouncing on a spring is a turntable rotating at a uniform speed. The plane of the turntable is vertical and perpendicular to this page. When lighted from in front, shadows of both the ball and a knob on the turntable are projected on a screen side by side. The speed of the turntable can be adjusted to make the two shadows move in unison.

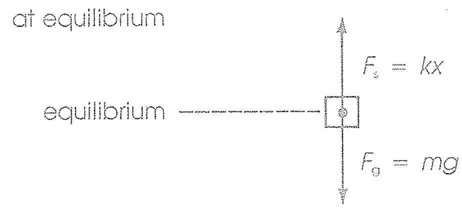


A side view of a ball bouncing on a spring and a rotating turntable, each with the same frequency. The shadow of the knob will be projected on the screen alongside the shadow of the ball.

While the knob on the turntable continues to move with uniform circular motion, its shadow, or projection, moves up and down along a line parallel to a diameter of the circular turntable. However, although the speed of the knob on the turntable is constant, the velocity of the shadow is constantly changing. At any instant, the velocity of the shadow, v, is equal to the vertical component of the velocity of the knob, v_0 (see illustration on next page).

At the midpoints, B and B′, the vertical component of the knob velocity is the same as the velocity of the knob itself, with the shadow moving at maximum velocity. This parallels the case for the spring as it moves through the equilibrium position. At the end points, A and C, the vertical component of the knob velocity is

The net force, F_{net}, acting upward on the mass contains one upward component, F_s, due to the spring, and one downward component, F_g, due to gravity.
At equilibrium, $F_{net} = F_s - F_g = 0$ and $F_s = F_g$, or $mg = kx$ where x is the stretch of the spring at the equilibrium position.



At a displacement Δx above the equilibrium position, the spring is shorter and

$$F_s = k(x - \Delta x)$$
$$= kx - k\Delta x$$

However, F_g is unchanged, so

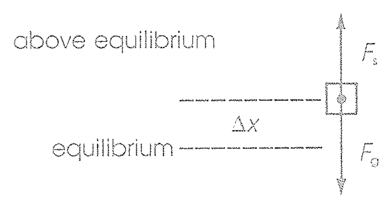
$$kx = mg$$

Therefore $F_s = mg - k\Delta x$

Since $F_{net} = F_s - F_g$

then $F_{net} = mg - k\Delta x - mg$

$$= -k\Delta x$$

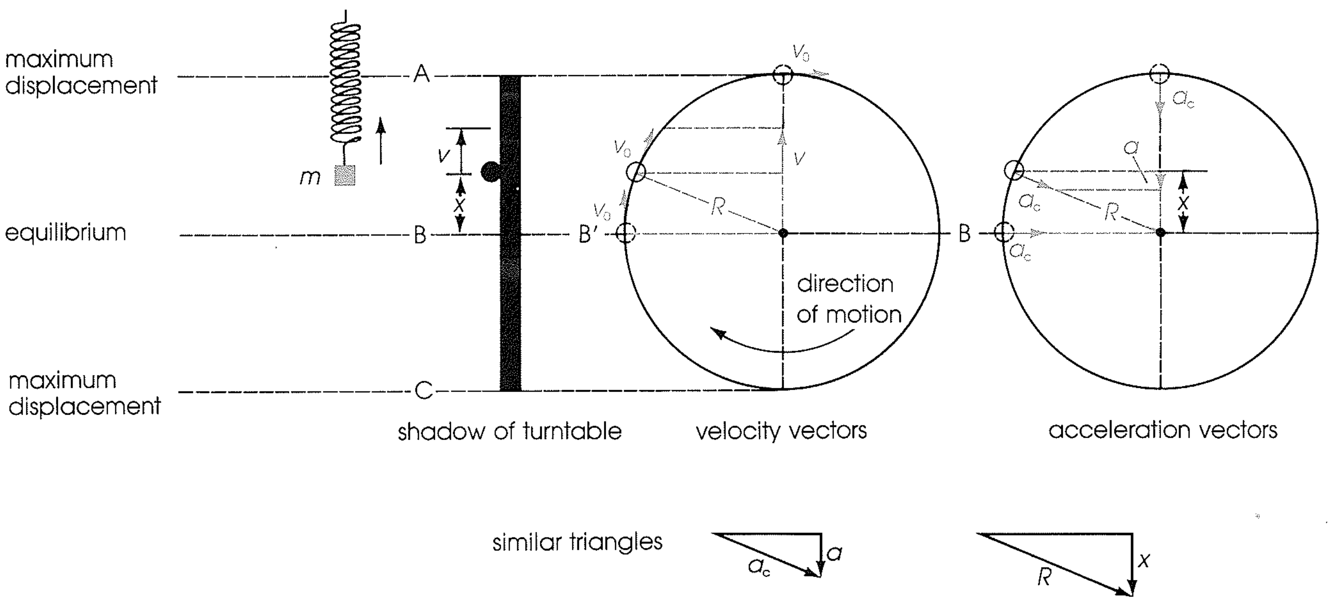


Above the equilibrium position, F_{net} acts downward to return the mass to the equilibrium position.
Similarly, for Δx below the equilibrium position,

$$F_{net} = +k\Delta x$$

Below the equilibrium position, F_{net} acts upward to return the mass to the equilibrium position.
So the spring and gravity combine to give a restoring force proportional to the displacement, and the constant is simply the spring constant.

zero, and the shadow is momentarily at rest. When the knob revolves in a circle, its shadow moves up and down with a velocity that varies from zero at the end points to a maximum at the midpoint. The shadow projected on the screen exhibits the same simple harmonic motion as the bouncing ball on the spring.



To further demonstrate that the motion of the shadow is S.H.M., a reference circle has been drawn to the right of the velocity circle to show the acceleration of the knob. Since there is uniform circular motion, the knob will have a constant centripetal acceleration, a_c of constant magnitude. At any instant, the acceleration of the shadow, a is equal to the vertical component of the centripetal acceleration of the knob. Thus, at the midpoint the acceleration of the shadow is zero, while at the end points it is at a maximum — downward when at the top and upward when at the bottom. In all cases the acceleration of the shadow acts towards the midpoint, which is a characteristic property of S.H.M.

Examining the acceleration reference circle, we can see from the similar triangles that $\frac{a}{x} = \frac{a_c}{R}$ where x is the vertical displacement from the equilibrium position. But, a_c and R are constant throughout the motion, that is, $\frac{a_c}{R}$ = constant; therefore $\frac{a}{x}$ = constant, or $a \propto x$. The acceleration is directly proportional to the displacement.

This is a general property of all S.H.M.

The centripetal acceleration for an object moving with a uniform speed in a circle is given by $a_c = \frac{4\pi^2 R}{T^2}$ (Section 3.9). Therefore the ratio a_c/R reduces to:

$$\frac{a_c}{R} = \frac{4\pi^2 R/T^2}{R} = \frac{4\pi^2}{T^2}$$

But since $\quad \frac{a}{x} = \frac{a_c}{R}$

then $\quad \frac{a}{x} = \frac{4\pi^2}{T^2}$ where T is the period of oscillation.

This relationship is true for all forms of simple harmonic motion; the ratio $\frac{a}{x}$ is always $\frac{4\pi^2}{T^2}$. When solved for T, this will give an expression for the period of any simple harmonic motion.

$$T = 2\pi \sqrt{\frac{x}{a}}$$

where T is the period, in seconds
x is the displacement from equilibrium, in metres
a is the acceleration, in metres/second squared

The accelerations of the shadow, a, and the knob, a_c, are equal in magnitude when x is at its maximum value. Thus, in this equation x can be replaced by the amplitude, A, provided a represents the maximum acceleration of the object undergoing simple harmonic motion.

Now we shall apply this relationship to two examples of simple harmonic motion, the period of a mass hung from a spring and the period of a simple pendulum.

Period of a Mass Hung from a Spring

A mass is supported by an ideal spring that obeys Hooke's Law, i.e., $F = kx$. If the mass undergoes an acceleration, Newton's Second Law applies, i.e., $F = ma$. At the end points of the oscillation, where x equals the amplitude, the acceleration is a maximum and can be determined as follows:

$$ma = kx$$

$$a = \left(\frac{k}{m}\right) x$$

Substituting in $$T = 2\pi\sqrt{\frac{x}{a}}$$

$$T = 2\pi\sqrt{\frac{x}{\left(\frac{k}{m}\right)x}}$$

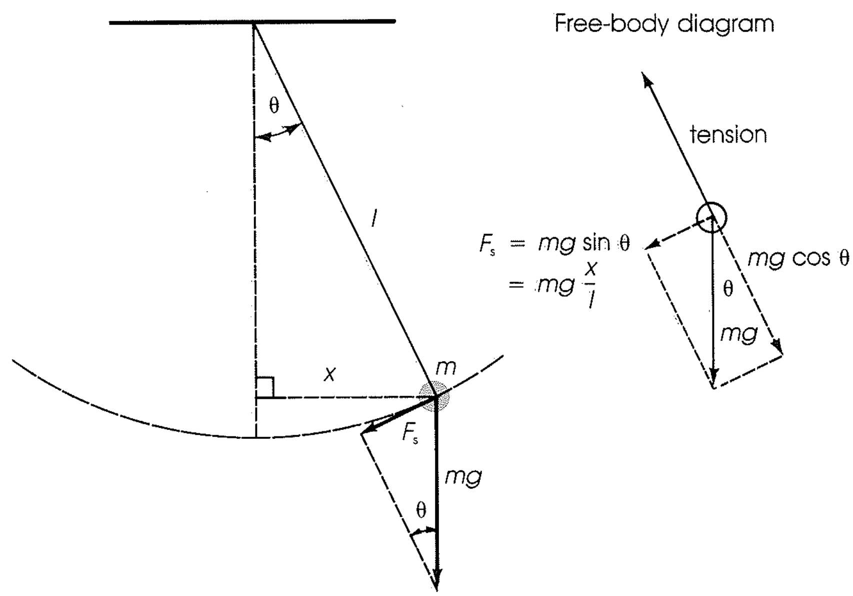
$$\mathbf{T = 2\pi\sqrt{\frac{m}{k}}}$$

This derivation for the period of the mass of the spring does not include the analysis described in the marginal note on page 449. However, gravity "cancels out", as shown there.

This equation tells us that where k is large (as for a stiff spring) the period of vibration is small. It also shows that a larger mass vibrates more slowly: when m is larger, T is larger. Note that the period of the vibration does not depend on its amplitude. It makes no difference whether the body vibrates over a large distance or a small one. The period remains the same, provided the amplitude does not exceed the elastic limit of the spring.

Period of a Simple Pendulum

A simple pendulum consists of a mass, usually called a pendulum bob, suspended from the end of a light string. We assume that the mass of the string is negligible, and ignore it in our analysis.



As the pendulum swings back and forth, the net force on the pendulum bob along the arc is F_s, the restoring force. F_s is the component of F_g that is tangent to the arc of motion, as illustrated. At any instant, the component of the weight (mg) in the direction of the tangent provides the restoring force.

The tension in the string has no component in the direction of the motion.

From the similar triangles we can write

$$\frac{F_s}{mg} = \frac{x}{l}$$

where l is the length of the pendulum and x is the horizontal displacement from the equilibrium position

Rewriting this equation, we have

$$F_s = \frac{mg}{l}x$$

Since mg and l are constants for a given pendulum, $F_s \propto x$. But to be S.H.M., the restoring force would have to be proportional to the displacement of the bob *along the arc*, not to the horizontal displacement. For small angles of swing, that is, where θ is small, the distance x and the distance along the arc are very nearly equal in length. Thus we can conclude that the pendulum does exhibit S.H.M., but only for small angles of swing.

Since $F_s \propto x$, the relationship has the same form as $F \propto x$, or $F = kx$ (Hooke's Law).

$$F_s = \frac{mg}{l}x$$

$$kx = \frac{mg}{l}x$$

and

$$k = \frac{mg}{l}$$

Substituting in $T = 2\pi\sqrt{\frac{m}{k}}$ (from the spring example)

$$T = 2\pi\sqrt{\frac{m}{\frac{mg}{l}}}$$

$$\mathbf{T = 2\pi\sqrt{\frac{l}{g}}}$$

It is important to note that this equation is only an approximation, but it is a valid one for small angles of swing. It is also important to note that neither the mass nor the amplitude appears in the final equation. Thus, the period of a pendulum is independent of both the mass and the amplitude of swing. On the other hand, since g appears in the equation, this equation can be used to determine the value for the acceleration of gravity.

Simple pendulums were used in the past to measure the strength of the Earth's gravitational field (g) at various points on the Earth's surface. Variations in gravity readings indicated the possible presence of iron and other heavy ores under the ground. Today, sensitive electronic gravitometers, pulled behind airplanes, have replaced the pendulum in this application.

S.H.M. is Sinusoidal

The sinusoidal nature can be illustrated by going back to the reference circle (see page 450). Since the circle is travelled once in time T, it follows that

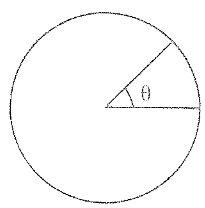
$$\frac{\theta}{2\pi} = \frac{t}{T}$$

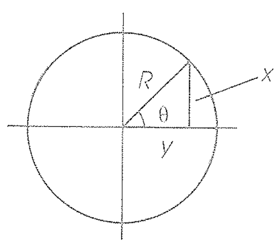
$$\text{or } \theta = \frac{2\pi t}{T}$$

$$x = R \sin \theta$$

$$= R \sin \frac{2\pi t}{T}$$

2π radians = 360° (one rotation)





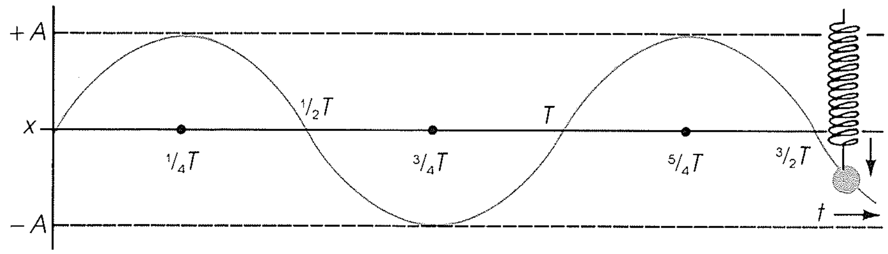
When $\sin \theta$ is a maximum ($\theta = 90°$), R is called the amplitude A.

Therefore, $x_{max} = A \sin \frac{2\pi t}{T}$ (as indicated in the text).

Another reason for studying sinusoidal waves is that any periodic wave of arbitrary form can be broken down into a set of sinusoidal components. This mathematical process is called Fourier analysis.

The phase angle is simply $2\pi \frac{t}{T}$ radians.

If a pen were attached to a vibrating mass on a spring, and a sheet of paper were moved at a constant rate behind it and in contact with the pen, a curve would be drawn as illustrated.



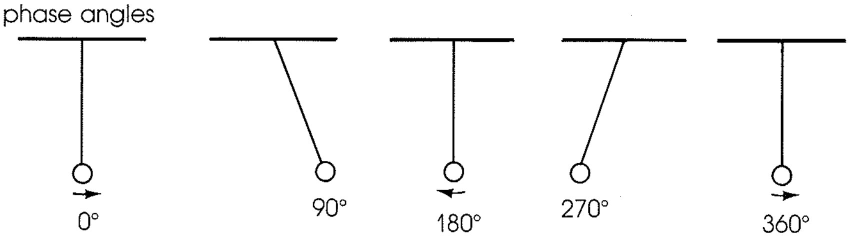
This curve has precisely the same shape as a sine curve, so called because it is like the graph of the sine function in trigonometry. Thus, the sinusoidal nature of S.H.M. can be expressed as

$$\mathbf{x = A \sin \frac{2\pi t}{T}}$$

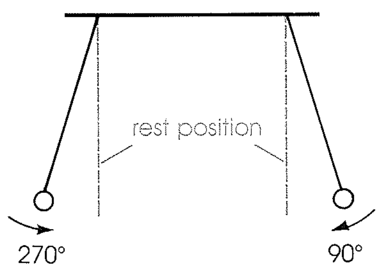
where x is the displacement from equilibrium
t is the time
T is the period
A is the amplitude

The sinusoidal function can be used to describe simple harmonic motion, using an angle instead of time as the variable. For example, in the case where a pendulum is at the centre, the angle is 0°. When it swings to maximum amplitude in one direction, the angle is 90°. When it returns to the equilibrium position, the angle is 180°. When it swings to maximum amplitude in the opposite direction, the angle is 270°, and when it returns to centre, the angle is 360°. These angles are called **phase angles**, and must not be confused with the angular position of the pendulum.

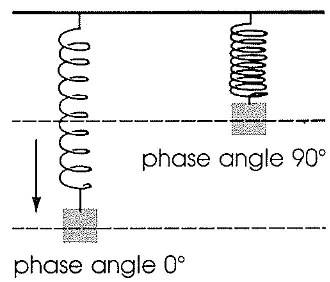
phase angles



If identical objects are vibrating **in phase**, with the same amplitude and frequency, at any given moment they have the same displacement from the rest position and are moving in the same direction and with the same speed. When this condition is not met, we say that the vibrating objects are vibrating **out of phase**. It is customary to describe the phase of two oscillating objects using phase angles. For example, if two identical objects are vibrating in phase, the difference in phase angle between them is zero. If they are vibrating out of phase, in such a way that they are equal distances from the rest position but travelling in opposite directions, we would say that the phase angle between them, or the phase difference, is 180° (see illustration). A phase difference of 360° means they are both in phase again.

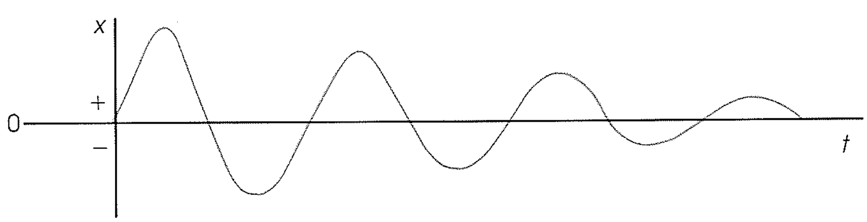


The pendulums are vibrating 180° out of phase.



The springs are vibrating 90° out of phase.

The amplitude of most vibrating objects slowly decreases with time until the oscillations stop completely. This is generally produced by air resistance and internal friction within the vibrating system. The lost energy, usually dissipated as heat, results in a decreased amplitude of oscillation. Such a phenomenon is known as a **damped harmonic motion**. A typical graph of displacement versus time for a damped harmonic motion is illustrated below.



These damped oscillating systems are the rule in everyday life, but we study (undamped) simple harmonic motion because it is much easier to analyse mathematically. Except where damping is large, there is little effect on the period of vibration, and the equations for simple harmonic motion can be used.

Sample problems

1. A 1.0 kg mass is hung from a spring whose constant, k, is 100 N/m. What is the period of oscillation of this mass when it is allowed to vibrate?

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{1.0\ \text{kg}}{100\ \text{N/m}}}$$
$$= 0.63\ \text{s}$$

2. What is the period of a pendulum 0.40 m long?

$$T = 2\pi\sqrt{\frac{l}{g}}$$
$$= 2\pi\sqrt{\frac{0.40\ \text{m}}{9.8\ \text{m/s}^2}}$$
$$= 1.3\ \text{s}$$

3. A 50 kg gymnast carefully steps onto a trampoline and finds that it is depressed vertically 30 cm when it supports her mass in equilibrium. Assuming the trampoline has negligible mass and that it vibrates with S.H.M., find the natural period of oscillation of the gymnast if her feet never leave the trampoline.

$$F = mg$$
$$= (50\ \text{kg})\ (9.8\ \text{N/kg})$$
$$= 490\ \text{N}$$

This is the force exerted down upon the trampoline when she is standing on it.

The force constant for the trampoline is given by $F = kx$,

$$\text{so } k = \frac{F}{x}$$
$$= \frac{490\ \text{N}}{0.30\ \text{m}}$$
$$= 1.6 \times 10^3\ \text{N/m}$$

For S.H.M.,

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{50\ \text{kg}}{1.6 \times 10^3\ \text{N/m}}}$$
$$= 1.1\ \text{s}$$

4. A pendulum vibrates with a period of 2.0 s and an amplitude of 50 cm. What will be its displacement from the equilibrium position 0.75 s after passing through it? 1.5 s after passing through it?

$$
\begin{aligned}
x_1 &= A \sin \frac{2\pi t}{T} \\
&= (0.50 \text{ m}) \sin \frac{360^\circ (0.75 \text{ s})}{2.0 \text{ s}} \\
&= (0.50 \text{ m}) \sin 135^\circ \\
&= 0.35 \text{ m}
\end{aligned}
$$

Note: The positive value of x_1 indicates that the pendulum is on the same side of the equilibrium position that it entered as the problem began.

$$
\begin{aligned}
x_2 &= (0.50 \text{ m}) \sin \frac{360^\circ (1.5 \text{ s})}{2.0 \text{ s}} \\
&= 0.50 \sin 270^\circ \\
&= -0.50 \text{ m}
\end{aligned}
$$

The negative value of x_2 indicates that the pendulum has moved to the opposite side of the equilibrium position.

Practice

1. Calculate the period for a spring whose force constant is 15 N/m, if the mass on the spring is 1.0 kg. (1.6 s)
2. What is the period of a pendulum suspended from the CN tower in Toronto by a light string 4.96×10^2 m long? (44.7 s)
3. You are designing a pendulum clock. How far must the centre of mass of the simple pendulum be located from the pivot point of rotation to give the pendulum a period of 1.0 s? (25 cm)
4. A 2.5 kg object, vibrating with simple harmonic motion, has a frequency of 1.0 Hz and an amplitude of 0.50 m. What is the restoring force on the object at the ends of the swing? (49 N)
5. A 0.020 kg cart is held between two identical, stretched springs on the air track illustrated. A force of 2.0 N is employed to hold the cart in a position 0.10 m from equilibrium. The cart is then released and allowed to vibrate from the 0.10 m position.
   (a) What is the force constant for the springs/cart system?
   (b) What is the frequency of vibration?
   (c) What is the maximum kinetic energy of the cart?
   (d) Where does (c) occur?
   (e) What is the cart speed in (c)?
   (20 N/m, 5.0 Hz, 0.10 J, 3.2 m/s)